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Fuel Optimality of Cruise

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Introduction

IN the paper by Schultz and Zagalsky,[1] the solution characteristics for the minimum fuel-fixed range problem are determined for a number of different mathematical models of aircraft dynamics. Different results are obtained for different sets of equations. The energy state equations are shown to not allow a partial throttle, constant velocity, cruise solution; but because the velocity set is not convex, to have a "chattering" solution which provides the best performance but which may not be realizable with piecewise continuous maximum values of the controls. The equation set with throttle and flight path angle as controls was shown to allow a partial throttle cruise solution. This conclusion was shown to be invalid by Speyer[3] by application of the Generalized Legendre-Clebsch condition.

The following analysis shows that for a higher order set of equations which has lift and thrust as controls, the necessary condition for optimization including the Generalized Legendre-Clebsch condition are satisfied at the cruise point so that the partial throttle cruise condition is a candidate solution for the minimum fuel-fixed range problem.

Problem Formulation

The problem is to find the aircraft trajectory from an initial velocity, altitude, and range to a final velocity, altitude and range using minimum fuel. Mathematically the problem is to minimize the fuel used G,

$$G = \int_{t_o}^{t_f} \sigma T dt \tag{1}$$

Subject to the constraints:

$$\begin{aligned} \mathring{E} &= (T - D)V/M;\ E(t_o) = E_o;\ E(t_f) = E_f \\ \mathring{\gamma} &= (L - W)/MV \\ \mathring{h} &= V \sin\gamma;\ h(t_o) = h_o;\ h(t_f) = h_f \\ \mathring{x} &= V \cos\gamma;\ x(t_o) = x_o;\ x(t_f) = x_f \end{aligned} \tag{2}$$

where $E = (V^2/2) + gh$ is the specific energy, γ is the flight path angle, x is the range, σ is the specific fuel consumption assumed to be constant, M is the mass, V is the velocity, x is the range, and D is the drag. Drag is given by

$$D = QSC_{d_0}(M) + (KL^2/QS) \tag{3}$$

The control variables are thrust (T) and Lift (L).

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Problem Solution

The solution to the stated problem can be found by applying the maximum principle which states: The controls are determined from

$$\min_{u \in U} H \tag{4}$$

where

$$H = \sigma T + \lambda_1(T - D)V/M + \lambda_2(L - W)/MV^2 + \lambda_3 V \sin\gamma + \lambda_4 V \cos\gamma$$

$$\mathring{\lambda}_1 = -\frac{\partial H}{\partial E}, \quad \mathring{\lambda}_2 = -\frac{\partial H}{\partial \gamma}, \quad \mathring{\lambda}_3 = -\frac{\partial H}{\partial h}, \quad \mathring{\lambda}_4 = -\frac{\partial H}{\partial x} \tag{5}$$

and

$$\frac{\partial H}{\partial t} + \frac{dH}{dt} = 0, \qquad H dt_f - \lambda^T dx_f = 0 \tag{6}$$

Robbins[2] has shown that the solution must also satisfy the Generalized Legendre-Clebsch condition which apply to nonsingular and singular arcs. The Generalized Legendre-Clebsch condition is

$$\frac{\partial}{\partial u} \frac{d^q}{dt^q}\left(\frac{\partial H}{\partial u}\right) = 0 \quad \text{for all } t \text{ on the singular arc and } q \text{ odd}$$
$$(-1)^p \frac{\partial}{\partial u}\left[\frac{d^{2p}}{dt^{2p}} \frac{\partial H}{\partial u}\right] \geq 0 \quad \text{for all } t \text{ on the singular arc} \tag{7}$$
$$m = 2p$$

According to Robbins, m is the first value where the derivatives of $(d^m/dt^m)(\partial H/\partial u)$ do not vanish identically; i.e., controls can be determined from this condition

$$\frac{d^m}{dt^m}\left(\frac{\partial H}{\partial u}\right) = W_m(x, \lambda, t) + Q_m(x, \lambda, t)u \tag{8}$$

For multiple control variables, the arc may be singular to degree m $w{\cdot}r{\cdot}t$ some of the control variables and to some higher degree with respect to other control variables. By use of Eq. (8), r, of the control variables can be expressed as a function of x, λ, and t and the remaining ("more singular") control variables.

Substitution into the Hamiltonian gives a new Hamiltonian with fewer control variables. The new Hamiltonian and its first m-1 derivatives vanish identically. The mth derivative does not involve control variables but some higher degree derivatives will (in general). This gives a relation like Eq. (8) but with a larger m and a new matrix Q_m of smaller size. If the matrix is nonsingular, the process terminates.

Test of Cruise Solution

We now test the cruise solution:

$$L = W;\ T = D \qquad V = \text{const.};\ h = \text{const.} \tag{9}$$

to determine if the Maximal Principle and the Legendre-Clebsch condition are satisfied. Taking the first partial of H $w{\cdot}r{\cdot}t$ the controls:

$$(\partial H/\partial T) = \sigma + \lambda_1 V/M, \qquad \frac{\partial H}{\partial L} = -\lambda_1\left(\frac{\partial D}{\partial L}\right)\frac{V}{M} + \frac{\lambda_2}{MV} \tag{10}$$

The second of Eq. (10) can be solved for the lift:

$$L = \frac{\lambda_2 \rho S}{4\lambda_1 K} \tag{11}$$

Taking matrix of second partials of H

$$\begin{vmatrix} \frac{\partial^2 H}{\partial T^2} & \frac{\partial^2 H}{\partial T \partial L} \\ \frac{\partial^2 H}{\partial L \partial T} & \frac{\partial^2 H}{\partial L^2} \end{vmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{2\sigma K}{QS} \end{bmatrix} \tag{12}$$

Thus the problem is singular of degree zero with respect to L and of higher degree with respect to T. Substituting L from Eq. (11) into H and calling new H, H^*

$$H^* = \sigma T + \lambda_1[T - D(E, h, \lambda_1, \lambda_2)]V/M + \lambda_2[L(\lambda_1, \lambda_2, E) - W]/MV \tag{13}$$

where H^* now is only a function of T. Taking first partial $w \cdot r \cdot t$ T

$$(\partial H^*/\partial T) = \sigma + \lambda_1 V/M \tag{14}$$

Taking time derivative

$$\frac{d}{dt}\frac{\partial H^*}{\partial T} = \mathring{\lambda}_1 V/M + \lambda_1 \mathring{V}/M$$
$$= \frac{1}{V}\left[\frac{\lambda_1 V}{M}\left(\frac{\partial D}{\partial V}\right)_h + \lambda_2 \frac{(L - W)}{MV^2} - \lambda_3 \sin\gamma - \lambda_4 \cos\gamma + \lambda_1 gV \sin\gamma\right] \tag{15}$$

Taking time derivative again

$$\frac{d^2 \partial H^*}{dt^2 \partial T} = \frac{1}{V}\left[-\sigma\frac{\partial}{\partial V}\left[\left(\frac{\partial D}{\partial V}\right)_h\right]_h \frac{(T - D)}{M} + \mathring{\lambda}_1\left[-\sigma\frac{\partial}{\partial L}\left(\frac{\partial D}{\partial V}\right)_h + \lambda_2\right]\frac{\partial L}{\partial \lambda_1} + \mathring{\lambda}_2\left[-\sigma\frac{\partial}{\partial L}\left(\frac{\partial D}{\partial V}\right)_h + \lambda_2\right]\frac{\partial L}{\partial \lambda_2}\right] \tag{16}$$

Note that $\lambda_2 = 0$. This follows from the fact that $\lambda_2 = 4\lambda_1 LK/\rho S$, $\lambda_1 = -\sigma V/M$ and assumption (9). Using Eqs. (9-11) the terms in Eq. (16) are

$$\frac{\partial L}{\partial \lambda_1} = \frac{gV}{\sigma}; \quad \frac{\partial L}{\partial \lambda_2} = \frac{-\rho S V}{4K\sigma M}$$
$$\frac{\partial D}{\partial L}\left(\frac{\partial D}{\partial V}\right)_h = -\frac{4KW}{QSV}; \quad \lambda_2 = -\frac{4\sigma KWM}{\rho SV} \tag{17}$$
$$\mathring{\lambda}_1 = (T - D)\sigma/V$$

$$\left(\frac{\partial^2 D}{\partial V^2}\right)_h = \frac{2QS}{V^2}C_{do} + \frac{4QS}{V}\left(\frac{\partial C_{do}}{\partial V}\right)_h + Q\left(\frac{\partial^2 C_{do}}{\partial V^2}\right)_h + \frac{6KW}{QSV^2}$$

Thus the second time derivative is:

$$\frac{d^2}{dt^2}\left(\frac{\partial H^*}{\partial T}\right) = -\frac{\sigma}{V}\left[\frac{2QSC_{do}}{V^2} + \frac{4QS\partial C_{do}}{V\,\partial V} + \frac{QS\partial^2 C_{do}}{\partial^2 V} + \frac{4KW^2}{QSV^2}\right]\frac{(T - D)}{M} = 0 \tag{18}$$

Thus, $T = D$ and

$$\frac{\partial}{\partial T}\frac{d^2}{dt^2}\frac{\partial H^*}{\partial T} = -\frac{\sigma}{V}\left[\frac{2QSC_{do}}{V^2} + \frac{4QS}{V}\frac{\partial C_{do}}{\partial V} + \frac{QS\partial^2 C_{do}}{\partial V^2} + \frac{4KW^2}{QSV^2}\right] \leq 0 \tag{19}$$

which satisfies the Generalized Legendre-Clebsch condition, if $C_{do}(M)$ is monotonically increasing $w \cdot r \cdot t$ V at the cruise point, which it is for military aircraft with subsonic cruise points. The cruise velocity and altitude are determined from:

$$\left(\frac{\partial H}{\partial E}\right)_h = -\lambda_1\frac{V}{M}\frac{\partial}{\partial E}(D)_h + \lambda_4\frac{\partial V}{\partial E} = 0$$
$$\left(\frac{\partial H}{\partial h}\right)_E = -\lambda_1\frac{V}{M}\frac{\partial}{\partial h}(D)_E + \lambda_4\frac{\partial V}{\partial h} = 0 \tag{20}$$
$$H = -\lambda_1 DV/M + \lambda_4 V = 0 = \sigma D + \lambda_4$$

This reduces to:

$$\frac{\partial}{\partial E}\left(\frac{D}{V}\right)_h = 0\ ;\ \frac{\partial}{\partial h}\left(\frac{D}{V}\right)_h = 0\ ;\ \text{or} \min_{h,V}\left(\frac{D}{V}\right)$$

Comparison with Other Models

The model with lift and thrust as controls thus allows continuous arcs of partial throttles while the energy state equation and the $V - \gamma$ equations do not. The different conclusions are possibly due to the fact that in the energy state and $V - \gamma$ equations $L = W$ is assumed while in the lift-thrust equations lifting arcs ($L \neq W$) are allowed. Thus, in the energy state equation fast changes in altitude can occur without changing the induced drag (drag due to lift) while in the lift-thrust equations large increases in altitude require large lift forces and correspondingly large induced drag forces.

Conclusions

For the minimum fuel-fixed range problem using a dynamical model with lifting trajectories ($L \neq W$), the cruise condition is shown to be a minimizing arc by applying the Maximal Principle including the Generalized Legendre-Clebsch condition for singular arcs. Other dynamical models do not allow a cruise solution but do allow a "chattering" cruise solution.

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Growing Procedural Problems of Washing Mammoth Aircraft

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Introduction

SIZE has always impressed Americans. American industry has augmented the philosophy of the super-size economy into creations of massive earth moving equipment, su-

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